Mathematics as Conversation: A Model for a Mathematics Retrieval Programme Conducted With Small Groups

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This paper describes a conversation in a small group where the teaching is based on a model for teaching the language arts. Students in the group are engaged in a non-routine task and talk about their ideas, revealing their understandings. Some of the students' responses are discussed, highlighting the ways in which linguistic variations can be used reveal students' conceptual understandings. It is suggested that such a model may be applicable to other mathematics retrieval classes.

Curriculum documents over the last decade have recognised that communication in mathematics is important, but may take different perspectives. For example, *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics (NCTM), 2000) adopts an explicitly broad view of communication, whilst *Mathematics Years 9-10 Syllabus* (Board of Studies NSW, 1996a, 1996b) describes the development of mathematical communication skills in terms of the mastery of the formal mathematical lexicon, with an explicit emphasis on written forms. Verbal communication is presented as playing a supporting role in the development of the written language of mathematics. If, however, mathematics is viewed as a language, the development of all aspects of mathematical communication becomes central to the development of students' mathematical understanding (Usiskin, 1996; Sfard, 2002). Far greater emphasis may be placed on students verbalising their ideas and on creating an environment where student-student and student-teacher interactions are fostered in the interests of deeper, more meaningful and connected learning.

This paper describes an approach to teaching that uses techniques of language-arts teaching together with non-routine tasks that generate student conversations in small groups. The conversations reveal student understandings of the mathematics associated with the tasks. One fragment of a discussion in a class conducted with a small group of students will be discussed in order to demonstrate how a conversational approach can reveal student understandings. The paper consists of: a summary of the theoretical underpinnings of the model used; a section describing the group of students, the task undertaken and which records and discusses parts of the conversation, and finally, a section which draws some conclusions and implications for teaching.

Background

Considerable research has been conducted on the many aspects of mathematical communication—the spoken, the written and the representational (e.g., Pimm, 1987; Ellerton & Clements, 1996; Bills & Grey, 2001; Bills, 2002). Pimm's (1987) observation, that, despite wide advocacy, classroom discussion of mathematics does not appear to have gained wide implementation, still holds for many classrooms today. If mathematics is viewed as a language (e.g., Pimm, 1987; Usiskin, 1996; Sfard, 2002), then communication lies at the centre and methods associated with the teaching of the language arts become appropriate strategies with which to teach mathematics. Such strategies include immersion in the environment of the language, contextualising the language, careful modelling by teachers and many opportunities for students to practise using the language. (Bickmore-Brand, 1993). In the teaching and learning model discussed in this paper, the context and interest are established by the use of non-routine tasks. The immersion is achieved by encouraging discussion whilst the teacher models appropriate language skills and encourages students to use clear, precise and meaningful language.

Two paradigms inform the teaching and learning model discussed in this paper. The first establishes the learning environment and is based on a model outlined by Bell (1993). The second suggests that students' use of language may indicate their conceptual understanding about the mathematics being discussed. This is based on research by Grey and Bills (2001) and Bills (2002) in identifying language structures of successful and unsuccessful students as they explain their procedures for performing mental calculations.

Bell (1993) used a model developed for teaching mathematics to ESL (English as a Second Language) students. However, it is a model that is particularly useful when working with students whose mathematical experiences have not lead to robust feelings of success. The model consists of "…10 carefully planned elements. The first five are instructional, and the second five are conversational" (Bell, 1993, p.149). The elements are: a thematic focus chosen by the teacher; the use of relevant student backgrounds and knowledge; direct teaching when necessary; the modelling and promotion of appropriate and technical language; the asking of questions such as: "How do we know…?", " Show me why…?"; the use of few "known answer" questions; acknowledging and responding to all student contributions; a discourse that builds on and connects ideas; a challenging, non-threatening environment; a general participation in a conversation that takes the direction dictated by the students rather than the teacher. This framework establishes the environment that encourages the conversations that provide the linguistic clues to student understandings.

Bills and Grey (2001) examined the responses of young students who, after performing a mental calculation were asked to describe what was in their head as they did the calculation. Their responses were categorised as *particular* when students simply described the calculation with specific numbers, *generic* when the numbers were used "as a vehicle to describe a procedure", and *general* when the students made little or no reference to specific numbers as they described the procedure. This work serves as a basis for an initial analysis of the conversation discussed below. In this paper, the terms "particular", "generic" and "general" are used in a similar way in the context of geometry. Bills (2002) further examined the students' descriptions of their mental calculations, concluding that different use of pronouns, tense and causal connectives such as "so", "if", "because" are associated with different levels of achievement (p.97). This study applied these categories and definitions to an analysis of a conversation about geometry between three senior high school students.

The learning and teaching model outlined emphasises the verbal aspects of mathematics communication in contrast to the more common emphasis on the written forms. Unlike other "natural" languages the written form in mathematics plays a "pre-eminent" role (Usiskin, 1996), and as a consequence there is a perception that the written form is the most powerful, and hence, often the only means whereby mathematics is done and communicated. This can be problematic if, as Frid (1993) had found, the written forms are taught without students coming to understand the meaning of the symbols. As a consequence, students may manipulate symbols without meaning, and students can then act only according to a set of ill-understood rules and procedures. However, when mathematics takes place in an environment that encourages "discourse", there is a sharing of understandings and meanings with a consequent clarification and modification of concepts among the students (Sfard, 2002).

"Discourse" seems to imply a degree of formality and expertise on the part of the participants in a discussion having a certain direction. Pimm (1987) prefers the term mathematical "talk" to convey the idea of a less directed discussion where students use a mix of informal and formal language. As suggested by the *Instructional Conversations* model of Tharp and Gillimore described by Bell (1993), the term "conversation" is used in this paper to convey a degree of equality between the participants (teacher included), the informality of the classroom and the free-ranging nature of the course of the conversations.

For students lacking confidence, speaking about their ideas may often be easier than writing (McGregor, 1993). Because mathematics is often taught principally in the written form, students failing to access the meaning of the symbols and mathematical writing conventions feel that they cannot do mathematics (Pimm, 1987). The struggle to conceptualise mathematics does not appear so obvious when students speak about their mathematics, particularly when the use of informal language is acceptable and the talk is with peers. In small groups, as in this case, student thoughts can be made public, and the group collectively assist with appropriate vocabulary and with interpretations of one another's faltering attempts and so act as mediators with the teacher. As students come to realise that they need more precise ways of conveying their meaning, they develop a vocabulary more in keeping with a "mathematical register" (Pimm, 1987), and one which helps to clarify their understanding. Students do, however, have to have something worthwhile to talk about.

In the teaching and learning model used in this study, non-routine questions that are "open", or "goal-free" have been used as the "thematic focus" to provoke reflective thought and develop higher order thinking in students so that their mathematical knowledge is deeper rather than shallow, conceptual rather than procedural (Ellerton & Clements, 1996). Such open or investigative tasks rely on students working in small groups to solve the problem. Small group discussion can be an appropriate strategy to facilitate comprehension "as students attempt to explore, investigate and solve problems together" (Ellerton & Clements, 1996). The efficacy of such an approach in classes of twenty or more students remains subject to debate. Gooding and Stacey (1993) caution that the effective development of mathematical communication and understanding cannot be guaranteed, particularly where the group involves weaker students and the teacher is not always present to monitor and encourage full participation. In this case, there is but one group and the teacher is available to provide guidance for students as well as challenge their understanding and encourage their independence. The teacher is also able to listen to the students' talk in order to identify linguistic variations that may indicate different conceptual development (Bills & Grey, 2001; Bills, 2002).

The students in the study report that their mathematical experiences have mostly been in traditional classrooms. Traditional classes tend to emphasise the use of the formal mathematical register, with all of its meaning-dense symbols and representations. In traditional classrooms, closed questions are the norm (Ellerton & Clements, 1996). The focus is on the students obtaining an answer, preferably using a standard procedure. However, when such a culture does not engender feelings of success, the students' may need to experience mathematics in other ways in order to promote more positive attitudes.

This study uses a teaching and learning model developed on the principles described above in order to achieve an affective change in the students. The analysis of the conversation, using the notion of linguistic pointers, demonstrates the potential such a model offers to those seeking to elucidate students' thinking. The analysis is, at this stage, speculative, but provides a starting point for further investigation.

The Study

The study involved a small after-school tutor group conducted by a community organisation in a rural town. The focus is on parts of a conversation between three students and their tutor (the author) during the first classroom session of the year. The students attend different schools at some distance from the town. There are two girls and one boy. One student, Mary (M), is in year 12, sitting the NSW HSC in Mathematics, which is predominantly an introductory calculus-based mathematics course (2U). The other students have just started year 11. Ann (A) is studying the General Mathematics course, which is based on algebra and arithmetic applications, and Tom (T) the Mathematics Extension 1 course (old 3U course), which incorporates the Mathematics (2U) and extends the calculus aspects.

The students attend the group because they are not confident in their ability to handle senior mathematics and want better grades. The primary aim of the tutorials is to foster a more positive attitude to mathematics and hence to increase the students' confidence as well as their understandings of the underlying concepts. The tutorial sessions need to strike a balance between the immediate needs of the students, to understand recent class work or to complete assignments, and the longer-term aim of establishing sound mathematical concepts. In order to do so, the students are presented with tasks that are, at least in part, non-routine. The tasks may have no unique solution, or may permit a variety of approaches to the solution. The first class of the year introduces the students to the types of tasks they will encounter. The tasks ought to be accessible at some level by each of the students and such that their responses will provide information about their mathematical strengths and weaknesses (Stoessiger & Edmunds, 1990).

The students were presented with five questions which addressed aspects of measurement, geometry, algebra, number and statistics. Only one question of the five forms the basis of the conversation. The conversation during the session was recorded as teacher notes and subsequently analysed for linguistic variations that might give clues as to the understandings of the students.

The Question

Consider rectangles with a fixed area of 36 square units. What are some possible rectangles? How many different rectangles can be made? Justify your answers in as much depth and detail as possible. (Adapted from *Principles and Standards for School Mathematics* (NCTM, 2000, p.228).)

The Conversation

The conversation about the above question began as the tutor (Tu) asked Ann for her responses to the question. Ann stated that she did not understand it. Mary tried to clarify the problem posed to Ann:

M: The angles are acute, obtuse…
T: It's about rectangles.
M: (After drawing several quadrilaterals) It doesn't make sense.
Tu: What is a rectangle?
M: A four-sided shape.
T: It's got four sides, four angles and two different length sides.

Tu: What is a trapezium?
M: It looks like…It has three different length sides.

The tutor interpreted this statement by drawing an isosceles trapezium, then presented Mary with several other trapezia in different orientations. Ann was then asked to read the question and she suggested one possible rectangle. This was sufficient for Mary to write down other rectangles having whole number dimensions. Tom agreed with Mary.

Tu: Any others?
T: You can't have 6 x 6. It's not a rectangle!
Tu: Isn't it?
M, A and T: (Not quite in unison) No. It's got all even sides. It's a square. A square isn't a rectangle.
Tu: Tell me, what are the characteristics of a rectangle?

Tom repeated his earlier statement. Mary added the fact that opposite sides were equal and Ann stated that all angles were right angles. The tutor, meanwhile had cut out a square and as the students listed the properties, pointed them out on the square. Silence. The image of a unique figure with all equal sides dominated. Tom was openly unconvinced.

Tu: What other quadrilaterals are there?
T: Rhombus, trapezium, parallelogram.
Tu: Is a rectangle a parallelogram?
T: Yes, but it has four right angles.
Tu:It does have four, but you really only need one to make a parallelogram a rectangle.
A: How come?

The tutor attempted a rough demonstration asking the students to form a "parallelogram" with their forearms, then, make a right angle at one vertex whilst keeping their arms parallel. Mary managed this successfully, but Tom and Ann focused only on making the right angle. Mary demonstrated, they mimicked her, but seemed unconvinced.

Later, the tutor prompted students to consider the diagonals of a rectangle. In order to have the students engage with the idea, they were instructed to carefully draw a rectangle. Ann had difficulty with this. The other two drew the rectangle, and the diagonals. Ann did not seem to know where to put the diagonals.

Tu: What can you tell me about the diagonals?
T: They intersect.
Tu: Where?
T: At the origin.
Tu: What do you mean, "the origin"?
T: Where it starts.
Tu: Where did you begin to draw your rectangle?
T: Here (pointing to the lower left vertex).
Tu: The origin can't be here then? (indicating the point of intersection of the diagonals)
T: The middle then.
Tu: The middle of what?
T: The rectangle.

At this point, the tutor pointed out that the diagonals intersected at their mutual mid-point and that the appropriate term was "bisect".

A Brief Analysis of the Conversation

Although the task presented to the students was designed to elicit a measurement and/or an algebraic response, the sidetrack into geometry provided the substance for the conversation and exposed the students' geometrical understandings with respect to quadrilaterals. The analysis of the conversation and of what it may reveal of student understanding focuses only on the linguistic aspects and what these could imply about the students' understandings. Using the categories of "particular", "generic" and "general" of Bills and Grey (2001), and Bills' (2002) identification of the ways in which students use "causal connectives", the following observations and inferences concerning the students' understanding can be made.

The first is that all three of the students can identify some properties of a rectangle, but not those that necessarily define a rectangle as distinct from other quadrilaterals. For example, Tom lists the properties, "four sides, four angles and two different length sides" and Mary simply states, "a four-sided shape". These phrases suggest that the students are using the concept of a rectangle as a "generic" example of the group of quadrilaterals (Bills & Grey, 2001). Tom and Mary could provide a list of some properties particular to rectangles, such as the figure having four right angles and equal opposite sides, but their responses did not indicate that they conceived any causal relationship between one property and another (Bills, 2002). It was as if the students were imagining a rectangle and then listing the "visible" properties.

The second observation that is that all students initially denied the fact that a square was a rectangle. There are two points to be made about the students' language —the use of simile and the use of the conditional "but".

The students' used the simile "A square is *like* a rectangle", rather than the statement "A square *is* a rectangle". Tom and Ann did agree that the square was *like* a rectangle (not recorded here as verbatim), but, because of its having all sides equal, it could not be a rectangle. The students struggled with the abstract notion of the set of squares being a sub-set of the set of rectangles. It is as if, as suggested above, the students were listing properties as they "saw" the figure. Dominating their reasoning was the "visible" fact of the square having all four sides equal, emphasised by the paper model, not two "different length sides". Consequently, the students perceived and so conceived a square as being distinct from a rectangle.

Tom was prepared to agree that a rectangle could be a parallelogram, although, this was qualified by his "Yes, but it has four right angles". Tom's use of "but" indicates some cognitive reservations about a rectangle actually being a parallelogram. Had he used "and" ("Yes, *and* it has four right angles"), one could infer that Tom conceptualised a rectangle as a figure possessing all the properties of a parallelogram *as well as* other particular properties.

A third observation to be made is Tom's use of the word "origin" to describe the intersection of the diagonals. Although he later defined "origin" as meaning the "start", he needed to be prompted to demonstrate exactly what that meant. Then he described the diagonals as "intersecting" in the middle of the rectangle, rather than at their mutual mid-point. Tom apparently had not heard the term "bisect".

An inference that could be made from this exchange is that Tom's geometrical understanding relies on his accessing an appropriate image. The use of the word "origin" suggests that the image accessed by Tom was that of the two intersecting coordinate axes. His use of the term "middle" may well be an extension of that image, although it also suggests his concept of "rectangle" overshadows his attention to the detail of the diagonals as having some properties of their own at the same time as existing within the larger figure. Tom seems to focus on one geometrical aspect at a time, as evidenced by his response to the demonstration of a parallelogram moving to a rectangle.

Mary, on the other hand, grasped the idea of the need for only one right angle in a parallelogram to create a rectangle—at an intuitive level. Both Mary and Tom, appear to be more easily convinced by a physical demonstration, than by any logical argument.

The fourth observation concerns Ann's inability to draw diagonals in the rectangle. This indicates that she may understand "diagonal" to mean "oblique" lines Pimm (1987, p.84). Another associated inference is that because Ann had to operate on the image in front of her, her rectangle had no diagonals drawn (she had to draw them in). She literally could not see them. There were no oblique lines for her to call "diagonal".

Ann, Tom and Mary should have had considerable experience of geometry in the years 7 to 10. The NSW Syllabuses for both the Advanced and Intermediate mathematics courses (which courses these students have studied in Years 9 and 10, prior to their attending the class under discussion) require students to have studied quadrilaterals, their properties, and apply their knowledge to solve problems (Board of Studies NSW, 1996a; 1996b.). Examples of activities and questions suggested in the syllabus would have provided learning experiences to develop the understandings discussed in this paper.

So, what could the outcomes of this examination of the students' use of language mean? One hypothesis is that the students see (literally) the rectangle as representative of all quadrilaterals. This is the "generic" use of the concept of rectangle as described by Bills and Grey (2001). If the students could conceive of a "generalised" (Bills & Grey, 2001) quadrilateral they could then perceive that different quadrilaterals share some common properties and yet also possess other, particular properties. This perception underpins the conception that rectangles are a particular instance of quadrilaterals. The same logic would then compel the students to the conclusion that a square was, indeed a rectangle. In other words, the students need to develop skills and concepts that enable them to *reason* abstractly about quadrilaterals.

Conclusion and Implications

The above example and analysis postulates how students' understandings of a set of geometrical concepts about quadrilaterals may be inferred from what they say and how they say it. These are speculative, but serve to demonstrate how a teaching and learning model based on the learning of mathematics as a language could be developed to identify students' strengths and weaknesses. In most classrooms, teachers cannot always focus so carefully on the conversation between students. This is the difficulty with group work highlighted by Gooding and Stacey (1993). Even when teachers do listen intently to what students say, they often do so in order to correct students, rather than dwell on how student errors reveal student cognitive development. This second goal is usually more easily carried out by analysis of written work. This too, may be problematic when students experience difficulty accessing the meaning of formal mathematical symbols and writing. Consequently, students may not be able to reveal their thinking.

In this initial study, however, there was opportunity to encourage and monitor and guide student talk. Although written records of their approach to the tasks were kept, these were not the focus although their methods of recording served to complement the students' verbal articulation of ideas.

Teaching mathematics as a language means that there is a focus on the students' construction of a lexicon of meaning. The meaning is derived from student experiences and modified as students communicate. Conversation, and then later written expression, is the essence of communication. Thinking provides the reason to communicate.

The small group, the lack of curriculum constraints, the informal setting and the focus on students articulating their thoughts and challenging each other, with the group conversations able to be monitored by the tutor, is a model that may serve to guide teachers helping struggling students. In particular, it provides for the creation of an environment where teachers can focus on the language constructs of students to expose student understandings in ways that student writing cannot.

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